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Characterization of WT-Spaces Whose Derivatives Form a WT-Space*

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Let U be a finite dimensional linear space of continuously differentiable functions, containing constants. We prove that U', the space of derivatives, is a WT-space iff U has a basis $\{u_0,..., u_{n-1}\}$ that is a complete WT-system with $u_0 \equiv 1$.

In this article we consider the problem of determining when the space of derivatives of a WT-space is itself a WT-space. This problem is solved by employing a concept introduced by Zielke [3, Chap. 8], that of oscillation spaces, which we define subsequently.

Let U be a linear space of real-valued functions defined on a real interval. We call U a WT-space (for "weak Tchebysheff") if U has a basis, a WT-system, $\{u_0,..., u_{n-1}\}$ such that for all $x_0 < \cdots < x_{n-1}$, $\det\{u_i(x_j)\}_{i,j=0}^{n-1} \geqslant 0$. U is a WT-space iff no element has more than $n-1$ sign changes, n being the dimension of U. If $\det\{u_i(x_j)\}_{i,j=0}^{n-1}$ is positive for all $x_0 < \cdots < x_{n-1}$ then we call $\{u_0,..., u_{n-1}\}$ a T-system. For these and related notions see [3].

A standard technique in dealing with continuous WT-systems is the method of smoothing [2, p. 40]. For continuous u we form the integral

$$u^\varepsilon(t) = \frac{1}{\sqrt{2\pi}\varepsilon} \int_a^b e^{-(t-x)^2/2\varepsilon^2} u(x)\, dx, \qquad \varepsilon > 0. \tag{1}$$

Then $u^\varepsilon \to u$ uniformly in (a, b) as $\varepsilon \downarrow 0$. By extrapolating u linearly outside $[a, b]$ and performing the integral over a slightly larger interval we can get uniform convergence of u^ε in all of $[a, b]$. The usefulness of this technique lies in the fact that if $\{u_0,..., u_{n-1}\}$ is a WT-system then $\{u_0^\varepsilon,..., u_{n-1}^\varepsilon\}$ is a T-

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system. Thus every continuous WT-system can be uniformly approximated by T-systems of analytic functions.

If $u_0 > 0$ then u_0^ε are bounded away from zero for $\varepsilon \geqslant 0$. If $u_0 \equiv 1$ then we may define for $\varepsilon > 0$

$$v_0^\varepsilon \equiv 1, \qquad v_i^\varepsilon = u_i^\varepsilon / u_0^\varepsilon \qquad (i = 1 \text{ to } n-1). \tag{2}$$

One easily checks that $v_i^\varepsilon \to u_i$ uniformly as $\varepsilon \downarrow 0$, that is, if $u_0 \equiv 1$ then we may approximate $\{u_0,..., u_{n-1}\}$ by T-systems $\{v_0^\varepsilon,..., v_{n-1}^\varepsilon\}$ for which $v_0^\varepsilon \equiv 1$ for all $\varepsilon > 0$.

(3) DEFINITION. $\{u_0,..., u_{n-1}\}$ is called a *complete* $(W)T$*-system* if $\{u_0,..., u_i\}$ is a $(W)T$-system for $i = 0$ to $n-1$.

If $\{u_0,..., u_{n-1}\}$ is a complete WT-system then $\{u_0^\varepsilon,..., u_{n-1}^\varepsilon\}$ defined as in (1) forms a complete T-system.

(4) DEFINITION. We will call a (*weak*) *Markov basis* a complete $(W)T$-system $\{u_0,..., u_{n-1}\}$ with $u_0 \equiv 1$.

(5) DEFINITION. A function f is said to have an *oscillation* of length k if there are points $x_1 < \cdots < x_k$ and $\varepsilon = \pm 1$ such that

$$\varepsilon(-1)^i (f(x_{i+1}) - f(x_i)) > 0 \qquad (i = 1 \text{ to } k-1).$$

(6) DEFINITION. An n-dimensional linear space U of real-valued functions is called an *oscillation space* if no $u \in U$ has an oscillation of length $n+1$.

The following theorem characterizes T-spaces that are oscillation spaces.

(7) THEOREM [3, Theorem 8.8]. *If U is a T-space with constants then U is an oscillation space iff U has a Markov basis.*

In fact, one can also prove the following theorem.

(8) THEOREM. *Every continuous oscillation space on $[a, b]$ contains constants.*

Proof. Let U be a continuous oscillation space; then U is clearly a WT-space. If $1 \notin U$ then by [1, Theorem 1] there is an element $u \in U$ such that $1 - u$ alternates n times—in other words, there are points $x_0 < \cdots < x_n$ and $\varepsilon = \pm 1$ such that $\varepsilon(-1)^i (1 - u(x_i)) = \max_{[a,b]} |1 - u(x)|$ $(i = 0$ to $n)$. But then u has an oscillation of length $n+1$ in $x_0,..., x_n$, a contradiction. Hence we must conclude that $1 \in U$. ∎

(9) THEOREM [3, Theorem 8.3]. *Every n-dimensional oscillation space with constants contains an $(n-1)$-dimensional oscillation space with constants.*

(10) COROLLARY. *Every continuous oscillation space has a weak Markov basis.*

Proof. This follows directly from the previous two theorems. ∎

We can now prove our first characterization theorem, about oscillation spaces.

(11) THEOREM. *A finite dimensional linear space of continuous, real-valued functions is an oscillation space iff it has a weak Markov basis.*

Proof. If U is an oscillation space then (10) implies that U has a weak Markov basis. Conversely, suppose that U has such a basis. We may then smooth and form the function v_i^ε as in (2). For each $\varepsilon > 0$, $\{v_0^\varepsilon, \ldots, v_{n-1}^\varepsilon\}$ is then a Markov basis and $v_i^\varepsilon \to u_i$ uniformly, where $\{u_0, \ldots, u_{n-1}\}$ is our original weak Markov basis for U. If some $u \in U$ had an oscillation of length $n+1$ then so too would u^ε for $\varepsilon > 0$ sufficiently small. But this contradicts (7); hence U is an oscillation space. ∎

(12) THEOREM. *Let U be a finite dimensional linear space of continuously differentiable functions, which contains constants. Then U', the space of derivatives, is a WT-space iff U is an oscillation space.*

Proof. Assume that U has dimension n; since U contains constants U' has dimension $n-1$. Now suppose that U' is a WT-space. If some $u \in U$ had an oscillation of length $n+1$, say, $x_1 < \cdots < x_{n+1}$ such that

$$(-1)^i(u(x_{i+1}) - u(x_i)) > 0 \qquad (i = 1 \text{ to } n),$$

then by the mean-value theorem we could find points $x_i < \eta_i < x_{i+1}$ such that $(-1)^i u'(\eta_i) > 0$ $(i = 1$ to $n)$. Thus u' has $n-1$ sign changes, contradicting the assumption that U' is a WT-space. Conversely, suppose that some $u' \in U'$ has $n-1$ sign changes. Then there are points $x_0 < \cdots < x_n$ such that, for some $\varepsilon = \pm 1$,

$$\varepsilon(-1)^i u'(x) \geqslant 0 \qquad \text{in} \quad (x_{i-1}, x_i) \; (i = 1 \text{ to } n)$$

and $u'(x) \neq 0$ on a subinterval of each (x_{i-1}, x_i). Hence the function

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(t)\, dt \in U$$

has an oscillation of length $n+1$ at $x_0, \ldots, x_n$ so that U is not an oscillation space. This completes the theorem. ∎

(13) Corollary. *Let U be as in* (12). *Then the following are equivalent.*

(1) *U is an oscillation space,*

(2) *U has a weak Markov basis,*

(3) *U' is a WT-space.*

We close with an example.

(14) Example. The following example, from Zielke [3, p. 44], is of a T-space with no Markov basis. It can be shown [4] that in this case no weak Markov basis exists either. We demonstrate that the space of derivatives is not a WT-space.

Let $u_0(t) \equiv 1$, $u_1(t) = t(1-t)$ and $u_2(t) = (1-t)(t^2-1)$. Then $U = sp\{u_0, u_1, u_2\}$ is a T-space (and hence a WT-space) on $[-1, 1]$. Zielke proves that U contains no 2-dimensional T-space and hence has no Markov basis. We have $U' = \{u_1', u_2'\}$, where $u_1'(t) = 1 - 2t$, $u_2'(t) = (1-t)(1+3t)$. If we choose $t_1 = 0$, $t_2 = 1/2$ then we get for u_1' and u_2'

$$U' \begin{pmatrix} 1 \,, & 2 \\ t_1 \,, & t_2 \end{pmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 5/4 \end{vmatrix} = 5/4 > 0,$$

while the choice $t_1 = -1$, $t_2 = 1$ yields $\left|\begin{smallmatrix} 3 & -1 \\ -4 & 0 \end{smallmatrix}\right| = -4 < 0$. Since the determinant is a continuous function of t_1 and t_2, it remains negative for $t_1 = -1 + \varepsilon$, $t_2 = 1 - \varepsilon$ for small $\varepsilon > 0$. Hence U' has no basis that is a WT-system on $(-1, 1)$; that is, U' is not a WT-space on $(-1, 1)$. ∎

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